Modified HW2: Single-Impulse Orbital Trajectories

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1 Introduction

Modern space travel has been enabled through the use of multi-stage rockets, where a large booster stage puts a smaller upper stage on a sub-orbital trajectory, and the upper stage lights a new set of engines that carries it onward to orbit.

Due to some imagined computational limits inherent to the CF312 computer lab, I do not have the luxury of multiple stages to work with. Instead, I am only able to simulate a massive cannon with the ability to adjust: the initial starting velocity, the mass of the cannonball, and the altitude it is fired from.

In the following, I will investigate how the required single-impulse velocity to reach a circular orbit changes with altitude, the ballistic coefficient of the object, and the planetary body's gravitational parameter.

2 Methods

The analytical solution to the velocity required for a circular orbit, derived in appendix B, is

$$v_{circ} = \sqrt{\frac{GM}{R}}$$

This is valid only in the absence of atmospheric drag. However, it provides a good starting point to numerically computing the critical velocity needed to reach a circular orbit in presence of drag.

Furthermore, again because of drag, a perfectly circular orbit is not possible without a sustained impulse constantly working against the force of drag. For the purposes of this analysis, an orbit will be considered complete when, after one full revolution around the planet, the ending radius is within 1,000 meters of the starting radius.

2.1 Equation of motion

For a body subject only to aerodynamic drag and gravity from a massive body, the equation of motion (derived in Appendix A) is

$$\ddot{\boldsymbol{r}} = -\frac{\mu}{||\boldsymbol{r}||^3}\boldsymbol{r} - \frac{1}{2m}C_D A\rho(\boldsymbol{r})||\dot{r}||\dot{\boldsymbol{r}}$$

where $\mu = GM$ is the planetary body's gravitational parameter and $\rho(r)$ is the altitude dependent atmospheric density. For the body in motion C_d is the coefficient of drag, and A is the cross-sectional area.

To test the effects of drag, we can define a ballistic coefficient

$$\beta = \frac{m}{C_D A}$$

so that the equation of motion becomes

$$\ddot{\boldsymbol{r}} = -\frac{\mu}{||\boldsymbol{r}||^3}\boldsymbol{r} - \frac{1}{\beta}\rho(\boldsymbol{r})||\dot{r}||\dot{\boldsymbol{r}}$$

2.2 Testing altitude

To test the effects of altitude on the velocity required to reach a circular orbit, I ran

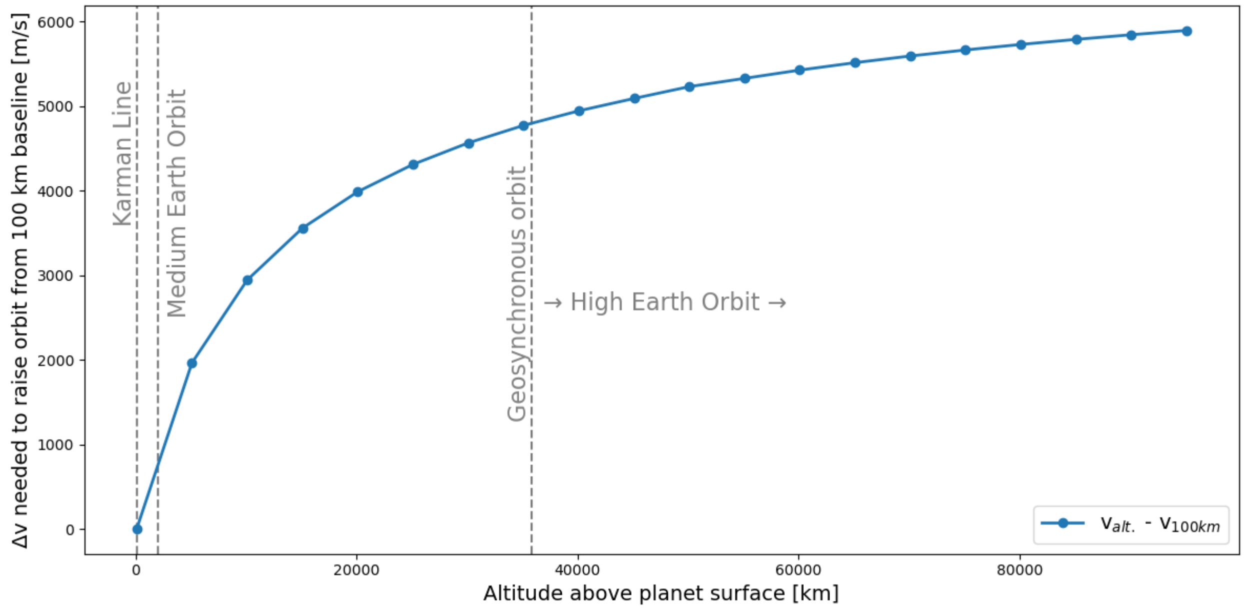

Figure 1: The velocity required to raise an orbit by 1000 km decreases as the initial altitude increases. Moving from 1,000 km to 2,000 km requires much more velocity change than moving from 50,000 km to 51,000 km.

a series of 20 trajectories. Each successive trajectory was spaced 1,000 km higher than the previous, between altitudes of 100 km and 100,000 km.

For each altitude, the required velocity was subtracted from the baseline altitude's required velocity to determine the extra velocity needed to reach the next circular orbit. I then plotted the altitude above surface vs. the required Δv to change orbits, seen in figure 1.

Atmospheric density was considered to be variable, and was calculated as

$$\rho(r) = \rho_0 e^{-r/H}$$

where r is the altitude above Earth surface, $\rho_0 = 1.225$ kg/m^3 is the atmospheric density at sea level, and $H = 8400$ m is the scale height for the atmosphere in meters.

The change in velocity for one altitude compared to the previous was calculated as

$$\Delta \mathrm{v} = \mathrm{v}_{\mathrm{alt}} - \mathrm{v}_{100\ \mathrm{km}}$$

2.3 Testing the ballistic coefficient

To analyze how the required velocity changes with the ballistic coefficient β, next I set a fixed altitude and tested a series of β values, where

$$\beta = \frac{m}{C_D A}$$

Leaving the coefficient of drag and the cross-sectional area the same, the mass was altered to change the ballistic coefficient in order to match the desired β.

The circular velocity in the absence of drag was then subtracted from the velocity required for each β value to show how much more velocity is needed.

$$\Delta \mathrm{v} = \mathrm{v}_\beta - \mathrm{v}_{\mathrm{vac}}$$

A series of 6 β values were tested, each an increment in order of magnitude, from $\beta = 1$ to $\beta = 100,000$.

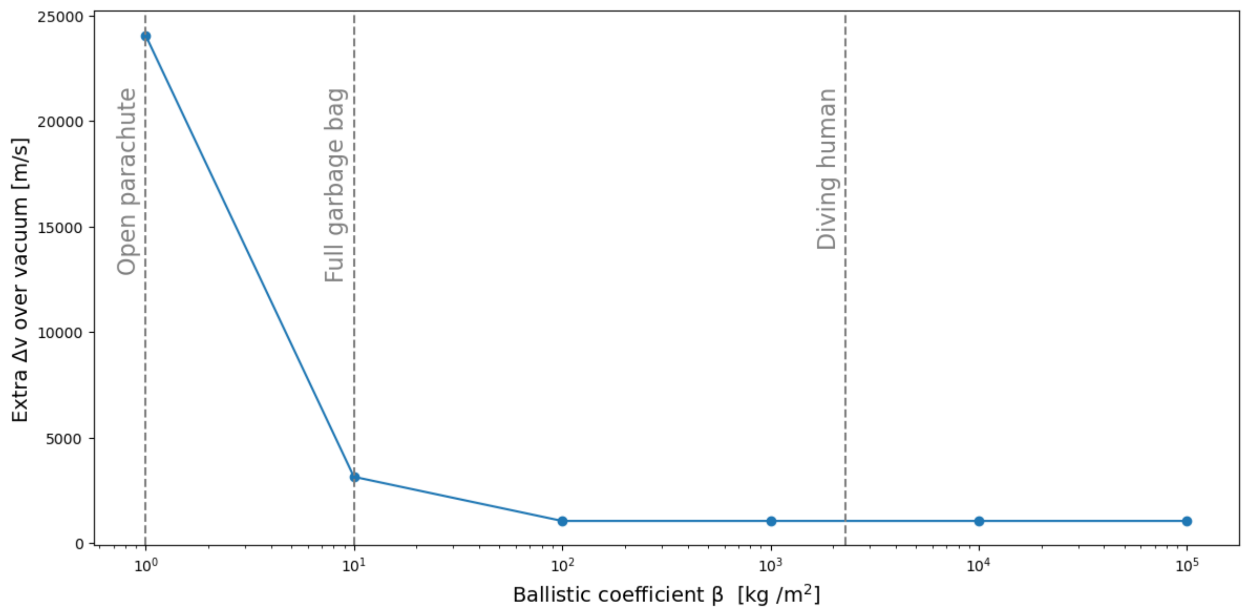

Figure 2: Comparing ballistic coefficients at an altitude of 100km shows significant decreases in the amount of velocity change needed for high-drag objects. Once an object is sufficiently low-drag, improving the ballistic coefficient yields negligible results.

2.4 Testing μ

The last parameter I tested was how the mass of the planetary body affects the required velocity. Similar to the previous two tests, I fixed the altitude and tested a series of μ values to observe how the required velocity changes with planetary mass, where

$$\mu = GM$$

with G as the gravitational constant, and M as the mass of the planet to be modified.

A series of 10 μ values were tested, from $\mu = 0.1$ to $\mu = 1000$. Each test was conducted at 100km, in the absence of drag. The only change to the parameters was the mass of the body – the radius was not changed.

The change in velocity was calculated as

$$\Delta \mathrm{v} = \mathrm{v}_{\text{100km, Earth } \mu} - \mathrm{v}_{\text{100km, new } \mu}$$

2.5 Brute-force sweep

For assessments of orbital trajectories for the altitude test, a brute-force sweep was applied to determine the minimum velocity needed to complete an orbit. To calculate the velocity needed to reach circular orbit around a planet, I began with computing the critical velocity using the analytical expression.

Using the analytical value as a minimum baseline for the necessary velocity, I first ran a number of trajectories within a range of twice the analytical expression. After one full revolution, an orbit, the final radius was subtracted from the initial radius.

A final radius less than the initial radius means the object was in a suborbital trajectory and lost too much energy due to drag. Once the final radius for a trajectory is within 1,000 m of the initial radius, the orbit is considered complete and that velocity is taken as the minimum required velocity for that orbit.

3 Results and Discussion

Planetary mass had the most significant impact on the velocity required to attain a circular orbit. For Earth, a single magnitude of increase in the mass increases the required ve-

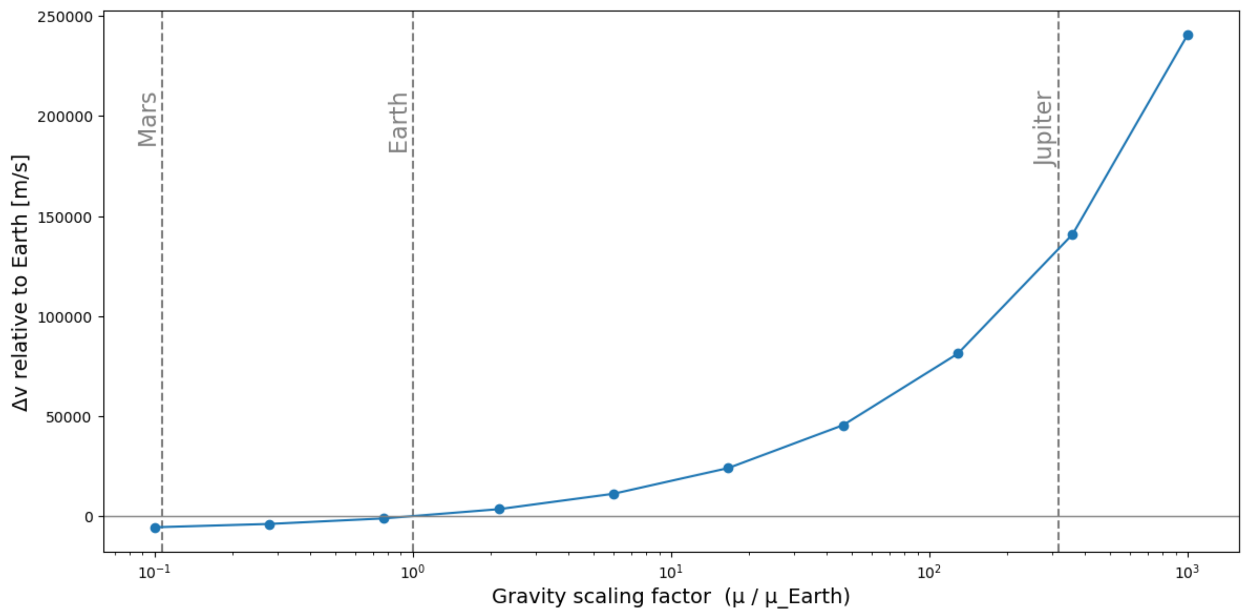

Figure 3: Changing the mass of the planetary body that an object is orbiting has a significant affect on the extra velocity change needed, relative to Earth at 100 km. If Earth was just one magnitude more massive, an object would require an extra 25,000 m/s of velocity to be in a circular orbit at 100km.

locity for 100km orbit by almost 25,000 m/s (figure 3).

To put this in perspective, the entirety of an Apollo mission's Saturn V rocket contained only about 15,000 m/s of Δv on the launchpad. To simply reach low-Earth orbit, a rocket launched from this "Super-Earth: would need just under two Saturn V's. Due to the limitations of chemical propulsion, achieving orbit quickly becomes impossible on more massive planets than Earth (lucky us, right?).

Next, changing the ballistic coefficient of an object can also have a significant impact on the velocity required to attain an orbit (figure 2). Even at 100km, where the atmospheric density is around a million times less dense than at sea level, the shape and mass of an object can be the difference between staying in orbit or crashing to Earth.

For a spacecraft that uses solar radiation for propulsion like the LightSail 2, deploying the boxing-ring sized solar sail in low-Earth orbit would have disastrous consequences on its orbital trajectory.

Lastly, the effects of the starting altitude also have a significant impact on the energy required to maintain an orbit. Raising an orbit from 1000 km to 2000 km requires an extraordinary amount of Δv compared to what would be needed to raise the orbit from 50,000 km to 51,000 km.

In real-world applications, this is a well-known physical consequence of orbital mechanics. In most rockets, around 85-95% of the mass is contained in the first stage, the part of the rocket that gets it up above the atmosphere. The second stage, in comparison, has a fuel load much smaller than the first stage, especially in low-Earth orbit applications.

This is why concepts such as orbital fuel depots have huge potential to advance our capability for interplanetary travel. The first stages of a rocket would boost the spacecraft up to a higher low-Earth orbit, where it would refuel at the orbital depot. From there, the requirements to raise the orbit are much less than getting to LEO, opening up many more potential destinations.

Appendix

A Derivation of the Equation of Motion

Starting from Newton's second law

$$\mathbf{F} = m\ddot{\mathbf{r}}$$

Where $\mathbf{r}$ is the position vector. For an object in Earth atmosphere, the force of gravity is

$$\mathbf{F}_{\mathrm{g}} = \frac{GMm}{r^2}$$

and we can label the gravitational parameter of the planet as $\mu = GM$. In vector form, the force is

$$\mathbf{F}_{\mathrm{g}} = \frac{\mu m}{||\mathbf{r}^3||}\mathbf{r}$$

For quadratic drag, the force on an object is

$$\mathbf{F}_{\mathrm{d}} = \frac{1}{2}\rho(\mathbf{r})C_D A||\mathbf{v}||\mathbf{v}$$

where $\rho(\mathbf{r})$ is the altitude-dependent atmospheric density, C_D is the coefficient of drag, A is the cross-sectional area of the object, and $\mathbf{v}$ is the velocity vector of the object.

The total force on the object is then

$$\mathbf{F}_{\mathrm{total}} = \mathbf{F}_{\mathrm{g}} + \mathbf{F}_{\mathrm{d}}$$

$$\mathbf{F}_{\mathrm{total}} = \frac{\mu m}{||\mathbf{r}^3||}\mathbf{r} + \frac{1}{2}\rho(\mathbf{r})C_D A||\mathbf{v}||\mathbf{v}$$

Dividing through by mass to yield the acceleration results in the equation of motion

$$\ddot{\mathbf{r}} = \frac{\mu}{||\mathbf{r}^3||}\mathbf{r} + \frac{1}{2m}\rho(\mathbf{r})C_D A||\mathbf{v}||\mathbf{v}$$

B Derivation of the velocity required for a circular orbit

In a perfectly circular orbit, at all times the force of gravity balances out the centripetal force,

$$\frac{GMm}{r^2} = \frac{mv^2}{r}$$

So the velocity required to be in such an orbit is simply

$$v = \sqrt{\frac{GM}{r}}$$